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A NOTE ON BIFURCATION POINTS OF TWO-PARAMETER MATRIX EIGENVALUE PROBLEMS

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Abstract. In this paper, two-parameter eigenvalue problems in terms matrix equations are considered. It is intended to report on a general framework on bifurcation points of the two-parameter matrix eigenvalue problems. Moreover, some results on partial derivative of matrix determinant involved in numerical scheme for computation of bifurcation points are also derived.

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1 Introduction

Two-parameter matrix eigenvalue problem consists of the following system of matrix pencils

$$\mathbb{P}_1(\lambda,\mu)u_1 := (M_{10} - \lambda M_{11} - \mu M_{12})u_1 = 0, \tag{1}$$
$$\mathbb{P}_2(\lambda,\mu)u_2 := (M_{20} - \lambda M_{21} - \mu M_{22})u_2 = 0, \tag{2}$$

where $0 \neq u_i \in \mathbb{R}^i$ and $M_{ij}, i := 1 : 2, j := 0 : 2$ are real matrices of order $n \times n$ for $i, j := 1 : 2$. The eigenvalues of this problem are the roots of the equations

$$f_i(\lambda,\mu) := \det(\mathbb{P}_i(\lambda,\mu)) = 0, i := 1 : 2 \tag{3}$$

that solves equations (1) and (2) simultaneously. Denote the problem by $\mathbb{W}$. Such a problem arise in diverse scientific fields (Plestenjak, 2017; Atkinson, 1972; Volkmer, 1988), and the references are therein. The origin of $\mathbb{W}$ can be found particularly in mathematical physics. It arises naturally in diverse scientfic domainsPons & Gutschmidt (2017); Attia & Barton (2021). It was (Atkinson, 1968, 1972), who first developed the theory for $\mathbb{W}$ with an unified approach on certain tensor product space. Researcher studied the spectral theory of $\mathbb{W}$ extensively over the years (Ghosh & Alam, 2018; Sleeman, 1978; Ringh & Jarlebring, 2021; Gil, 2016; Košir, 1994), elsewhere. A lot of numerical methods have also been developed so far to address the numerical solution of the problem $\mathbb{W}$ (Dong et al., 2016; Plestenjak, 2000; Rodriguez et al., 2018; Hochstenbach et al., 2005; Hochstenbach & Plestenjak, 2008; Ji, 1992).

The problem $\mathbb{W}$ can be converted into a single equation by considering the following matrices

$$M_0 = \begin{pmatrix} M_{10} & 0 \\ 0 & M_{20} \end{pmatrix}; M_1 = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{21} \end{pmatrix}; M_2 = \begin{pmatrix} M_{12} & 0 \\ 0 & M_{22} \end{pmatrix}$$

with order $n^2 \times n^2$. Then, in one equation the problem $\mathbb{W}$ reduces to

$$\mathbb{P}(\lambda, \mu)u := (M_0 - \lambda M_1 - \mu M_2)u = 0, \tag{4}$$

where $u := (u_1, u_2)^T$ and $\mathbb{P}(\lambda, \mu) := M_0 - \lambda M_1 - \mu M_2$ which depends linearly on the parameters $\lambda, \mu \in \mathbb{R}$. Denote the problem represented by the equation (4) by $\mathbb{P}$. The eigenvalues of $\mathbb{P}$ are the pair (λ, μ) that solves the equation (4) for some $u \neq 0$, called right eigenvector. Similarly, if there exist vector $0 \neq v$ such that $v^*\mathbb{P}(\lambda, \mu) = 0$, then v is called left eigenvector corresponding to the eigenvalue (λ, μ). Again, the eigenvalues of $\mathbb{P}$ can be obtained as the roots of the equation given by

$$f(\lambda, \mu) := \det(\mathbb{P}(\lambda, \mu)) = 0. \tag{5}$$

The equation (4) can be rewritten as

$$(M_0 - \mu M_2)u = \lambda M_1 u. \tag{6}$$

Equation (6) represents a generalized eigenvalue problem. For different values of μ, the solution of λ not always exist for the problem (6). But, if the matrix M_1 is invertible, then the equation (6) reduces to ordinary eigenvalue problem $M_1^{-1}(M_0 - \mu M_2)u = \lambda u$ that yield solution of λ for all values of spectral parameter μ, possibly complex. If all the matrices present in the matrix pencil (6) are of odd order, then by fundamental theorem of algebra there must exit at least one real root of $\lambda(\mu), \forall \mu$. Moreover, all solution of the problem (6) are real if matrices $M_i, i := 0 : 2$ are symmetric as well as M_1 is positive. Thus, if the problem $\mathbb{P}$ has a continuum solution, then numerical methods available for the generalized eigenvalue problem can be used to find simple eigenvalues for any fixed value of μ. After calculating different values of μ, the eigenvalue curves of problem $\mathbb{P}$ can be obtained on which the eigenvalue lies.

The aim of this paper, is to address the intersection points of the two system of families of eigencurves in the $\lambda - \mu$ plane. This point of intersection is called the bifurcation point of $\mathbb{P}$. Since the eigenvalue curves of the system (4) are algebraic functions, therefore they enjoys algebraic properties. Bifurcation points plays a crucial rule in some applied problems. This paper inspired by the paper Khlobystov & Podlevskyi (2009), where bifurcation points of simple eigenvalue curves of two-parameter problem have been calculated using iterative scheme. The scheme contains an efficient numerical procedure for the calculation of the derivative of matrix determinant based on LU decomposition of matrices.

Organization of the paper: In section 2, a review on some preliminaries shall be presented which will be used in the sections to follow. Section 3 contains a general framework on bifurcation point of two parameter problem and section 4 contains a concluding remarks on the whole work.

2 Preliminaries

Throughout the paper, we denote $adj(A)$ as the the adjoint of the matrix A, $tr(A)$ as the trace of the matrix A and dA as the differential of the matrix A respectively. $\sigma(\mathbb{P})$ shall represent the set of spectrum of the problem $\mathbb{P}$.

Definition 1. *Košir (1994). An eigenvalue (λ, μ) of the problem $\mathbb{P}$ is said to be simple if* $\dim(ker(\mathbb{P}(\lambda, \mu))) = 1$.

Proposition 1. *(Ghosh & Alam, 2018) Let M be any $m \times m$ matrix over $\mathbb{C}$ and $Rank(M) = m - 1$. Then $adj(M) = vu^*$ for some non zero vectors u and v such that $Mv = 0$ and $u^*M = 0$.*

Proposition 2. *((Ghosh & Alam, 2018), Jacobi formula) The mapping* $\det : \mathbb{C}^{m \times m} \to \mathbb{C}, A \to \det(A)$*, is differentiable and $ddet(A) = tr(adj(A)dA)$.*

3 Bifurcation points of Two-parameter eigenvalue problems

Consider the following finite set M consisting of points (λ, μ)

$$M := \Big\{(\lambda, \mu) : f(\lambda, \mu) = 0, f_{\lambda}^{'}(\lambda, \mu) = 0, f_{\mu}^{'}(\lambda, \mu) = 0\Big\} \tag{7}$$

such that $M \neq 0$. Then, there exist bifurcation points of the problem $\mathbb{P}$ (Khlobystov & Podlevskyi, 2009), i.e., the bifurcation points are solutions of the system of equations

$$\begin{aligned} [f(\lambda, \mu)]_{\lambda}^{'} &:= [\det(\mathbb{P}(\lambda, \mu))]_{\lambda}^{'} = 0 \\ [f(\lambda, \mu)]_{\mu}^{'} &:= [\det(\mathbb{P}(\lambda, \mu))]_{\mu}^{'} = 0 \end{aligned} \tag{8}$$

satisfying equation (5). Here the problem is to find the bifurcation points (λ^b, μ^b) as a solution of the system determinant equations (8). Various numerical algorithms have been developed Podlevskyi & Khlobystov (2010); Khlobystov & Podlevskyi (2009) to find bifurcation points, possibly the rough estimates of the same. In Khlobystov & Podlevskyi (2009), Khlobystov et. al., used Newton's method to find bifurcation points of eigenvalue curves under the agrees of the paper Podlevskyi (2007). Here, an overview on calculation of bifurcation points motivated by the papers Podlevskyi (2007, 2008) have been presented.

Assume that (λ_m, μ_m) be some approximations to the eigenvalue (λ, μ). Then the iterative scheme of Newton's method considered in Podlevskyi (2007) to the numerical solution of system (8) may be presented as

$$\lambda_{m+1} = \lambda_m + \Delta\lambda_m; \;\; \mu_{m+1} = \mu_m + \Delta\mu_m, \tag{9}$$

where $m = 1, 2, 3\ldots$ and the update $\Delta\lambda_m, \Delta\mu_m$ can be obtained by solving the following system of equations

$$\begin{aligned} [f(\lambda_m, \mu_m)]_{\lambda\lambda}^{''}\Delta\lambda_m + [f(\lambda_m, \mu_m)]_{\lambda\mu}^{''}\Delta\lambda_m &= -[f(\lambda_m, \mu_m)]_{\lambda}^{'}, \\ [f(\lambda_m, \mu_m)]_{\mu\lambda}^{''}\Delta\mu_m + [f(\lambda_m, \mu_m)]_{\mu\mu}^{''}\Delta\mu_m &= -[f(\lambda_m, \mu_m)]_{\mu}^{'}. \end{aligned} \tag{10}$$

System (10) can be written in the matrix form

$$A(\lambda_m, \mu_m)\Delta u = B, \tag{11}$$

where

$$A(\lambda_m, \mu_m) = \begin{pmatrix} [f(\lambda_m, \mu_m)]_{\lambda\lambda}^{''} & [f(\lambda_m, \mu_m)]_{\lambda\mu}^{''} \\ [f(\lambda_m, \mu_m)]_{\mu\lambda}^{''} & [f(\lambda_m, \mu_m)]_{\mu\mu}^{''} \end{pmatrix} \tag{12}$$

$$\Delta u = \begin{pmatrix} \Delta\lambda_m \\ \Delta\mu_m \end{pmatrix}; B = \begin{pmatrix} -[f(\lambda_m, \mu_m)]_{\lambda}^{'} \\ -[f(\lambda_m, \mu_m)]_{\mu}^{'} \end{pmatrix}.$$

Here, the determinant of the matrix $A(\lambda_m, \mu_m)$, whose elements are appropriately defined at the point (λ_m, μ_m), need to be nonzero. In each step of the iterative process it is necessary to calculate the value of the function $f(\lambda_m, \mu_m) = \det(\mathbb{P}(\lambda_m, \mu_m))$ and its partial derivatives only for the fixed values of the parameters λ and μ.

Generally authors uses the following Jacobi's formula based on trace of matrix to calculate the partial derivatives of matrix determinants involved in (10) and are given by

$$f_\lambda{}'(\lambda,\mu) = tr\left(adj(\mathbb{P}(\lambda,\mu))\frac{\partial}{\partial\lambda}[\mathbb{P}(\lambda,\mu)]\right) \tag{13}$$

$$f_\mu{}'(\lambda,\mu) = tr\left(adj(\mathbb{P}(\lambda,\mu))\frac{\partial}{\partial\mu}[\mathbb{P}(\lambda,\mu)]\right) \tag{14}$$

$$[f(\lambda,\mu)]_{\lambda\lambda}{}'' = \frac{\partial}{\partial\lambda}[tr\left(adj(\mathbb{P}(\lambda,\mu))\frac{\partial}{\partial\lambda}(\mathbb{P}(\lambda,\mu))\right)] \tag{15}$$

$$[f(\lambda,\mu)]_{\mu\mu}{}'' = \frac{\partial}{\partial\mu}[tr\left(adj(\mathbb{P}(\lambda,\mu))\frac{\partial}{\partial\mu}(\mathbb{P}(\lambda,\mu))\right)] \tag{16}$$

$$[f(\lambda,\mu)]_{\lambda\mu}{}'' = \frac{\partial}{\partial\lambda}[tr\left(adj(\mathbb{P}(\lambda,\mu))\frac{\partial}{\partial\mu}(\mathbb{P}(\lambda,\mu))\right)] \tag{17}$$

$$[f(\lambda,\mu)]_{\mu\lambda}{}'' = \frac{\partial}{\partial\mu}[tr\left(adj(\mathbb{P}(\lambda,\mu))\frac{\partial}{\partial\lambda}(\mathbb{P}(\lambda,\mu))\right)]. \tag{18}$$

Theorem 1. *If $(\lambda,\mu) \in \sigma(\mathbb{P})$ is any simple eigenvalue, then for some scalars $0 \neq \alpha, 0 \neq \beta$ there exist left eigenvector y and right eigenvector x of $\mathbb{P}$ such that*

1. $f_\lambda{}'(\lambda,\mu) = \bar{\alpha}\beta y^* \frac{\partial}{\partial\lambda}(\mathbb{P}(\lambda,\mu))x;$
2. $f_\mu{}'(\lambda,\mu) = \bar{\alpha}\beta y^* \frac{\partial}{\partial\mu}(\mathbb{P}(\lambda,\mu))x;$
3. $[f(\lambda,\mu)]_{\lambda\lambda}{}'' = \bar{\alpha}\beta y^* \frac{\partial^2}{\partial\lambda^2}(\mathbb{P}(\lambda,\mu))x;$
4. $[f(\lambda,\mu)]_{\mu\mu}{}'' = \bar{\alpha}\beta y^* \frac{\partial^2}{\partial\mu^2}(\mathbb{P}(\lambda,\mu))x;$
5. $[f(\lambda,\mu)]_{\lambda\mu}{}'' = \bar{\alpha}\beta y^* \frac{\partial^2}{\partial\lambda\partial\mu}(\mathbb{P}(\lambda,\mu))x;$
6. $[f(\lambda,\mu)]_{\mu\lambda}{}'' = \bar{\alpha}\beta y^* \frac{\partial^2}{\partial\mu\partial\lambda}(\mathbb{P}(\lambda,\mu))x.$

Proof. Let the eigenvalue (λ,μ) be simple. Then $Rank\ (\mathbb{P}(\lambda,\mu)) := n^2 - 1$. Thus, using proposition 2, there exist nonzero vectors $u, v \in \mathbb{C}^n$ such that $adj(\mathbb{P}(\lambda,\mu)) := vu^*$ with $\mathbb{P}(\lambda,\mu)v = 0$ and $u^*\mathbb{P}(\lambda,\mu) = 0$ for $i, j := 1:2$. Hence

$$tr\left(adj(\mathbb{P}(\lambda,\mu))\frac{\partial}{\partial\lambda}\mathbb{P}(\lambda,\mu)\right) := tr\left(vu^*\frac{\partial}{\partial\lambda}\mathbb{P}(\lambda,\mu)\right) = u^*\frac{\partial}{\partial\lambda}\mathbb{P}(\lambda,\mu)v. \tag{19}$$

Since eigenvalue is simple. So $Dim\,(Ker(\mathbb{P}(\lambda,\mu))) = 1$ for $i := 1:2$. So there exist scalars $0 \neq \alpha, 0 \neq \beta$ such that $u = \alpha y$ and $v = \beta x$ for $i := 1:2$. So

$$tr\left(adj(\mathbb{P}(\lambda,\mu))\frac{\partial}{\partial\lambda}\mathbb{P}(\lambda,\mu)\right) = (\alpha y)^*\frac{\partial}{\partial\lambda}\mathbb{P}(\lambda,\mu)\beta x = \bar{\alpha}\beta y^*\frac{\partial}{\partial\lambda}\mathbb{P}(\lambda,\mu)x. \tag{20}$$

Proof of (i) follows from equations (20) and (13). Similarly, (ii) can be proved by considering the derivative w.r.t μ. Again,

$$\begin{aligned}\frac{\partial}{\partial\lambda}\left(tr\left(adj(\mathbb{P}(\lambda,\mu))\tfrac{\partial}{\partial\lambda}\mathbb{P}(\lambda,\mu)\right)\right) &= \frac{\partial}{\partial\lambda}\left(u^*\tfrac{\partial}{\partial\lambda}\mathbb{P}(\lambda,\mu)v\right)\\ &= u^*\frac{\partial^2}{\partial\lambda^2}\mathbb{P}(\lambda,\mu)v\\ &= \bar{\alpha}\beta y^*\frac{\partial^2}{\partial\lambda^2}\mathbb{P}(\lambda,\mu)x.\end{aligned}$$

Thus,

$$[f(\lambda,\mu)]_{\lambda\lambda}'' := \bar{\alpha}\beta y^*\frac{\partial^2}{\partial\lambda^2}\mathbb{P}(\lambda,\mu)x. \tag{21}$$

Proof of (iii) follows automatically from equations (21) and (13). Equations (iv), (v) and (vii) can be proved in a similar manner. □

Theorem 2. *Let the matrix*

$$\begin{pmatrix} y^* \frac{\partial^2}{\partial \lambda^2}\mathbb{P}(\lambda,\mu)x & y^* \frac{\partial^2}{\partial \lambda \mu}\mathbb{P}(\lambda,\mu)x \\ y^* \frac{\partial^2}{\partial \mu \lambda}\mathbb{P}(\lambda,\mu)x & y^* \frac{\partial^2}{\partial \mu^2}\mathbb{P}(\lambda,\mu)x \end{pmatrix} \tag{22}$$

is nonsingular with y_m and x_m as the corresponding approximate left and right at the simple eigenvalue (λ_m, μ_m). Then the coefficient matrix $A(\lambda_m, \mu_m)$ defined in (12) is nonsingular for simple eigenvalue.

Proof. Let x_m and y_m be the m^{th} approximations of right and left eigenvectors corresponding to the eigenvalue (λ_m, μ_m). Using Theorem 1, the coefficient matrix $A(\lambda_m, \mu_m)$ defined in (12) at (λ_m, μ_m) reduces to

$$A(\lambda_m,\mu_m) = \begin{pmatrix} \bar{\alpha}\beta y_m^* \frac{\partial^2}{\partial \lambda^2}\mathbb{P}(\lambda_m,\mu_m)x_m & \bar{\alpha}\beta y_m^* \frac{\partial^2}{\partial \lambda \mu}\mathbb{P}(\lambda_m,\mu_m)x_m \\ \bar{\alpha}\beta y^* \frac{\partial^2}{\partial \mu \lambda}\mathbb{P}(\lambda_m,\mu_m)x_m & \bar{\alpha}\beta y_m^* \frac{\partial^2}{\partial \mu^2}\mathbb{P}(\lambda_m,\mu_m)x_m \end{pmatrix}$$

$$A(\lambda_m,\mu_m) = dai(\bar{\alpha}\beta, \bar{\alpha}\beta) \begin{pmatrix} y_m^* \frac{\partial^2}{\partial \lambda^2}\mathbb{P}(\lambda_m,\mu_m)x_m & y_m^* \frac{\partial^2}{\partial \lambda \mu}\mathbb{P}(\lambda_m,\mu_m)x_m, \\ y_m^* \frac{\partial^2}{\partial \mu \lambda}\mathbb{P}(\lambda_m,\mu_m)x_m & y_m^* \frac{\partial^2}{\partial \mu^2}\mathbb{P}(\lambda_m,\mu_m)x_m \end{pmatrix}. \tag{23}$$

Thus the nonsingularity of $A(\lambda_m, \mu_m)$ depends on the nonsingularity of the left hand side matrix in the equation (23). Hence the theorem. □

Example: (Khlobystov & Podlevskyi, 2009) Consider the homogeneous vertical beam under the influence of a vertical load. If the ends of the beam are restrained, then the problem is modeled into a spectral problem represented by the following fourth order differential equation

$$y^{(iv)}(x) + 2\mu y^{''}(x) - \lambda y(x) = 0, \tag{24}$$

where $x \in [0,1]$ with boundary conditions $y(0) = y^{'}(0) = y(1) = y^{'}(1) = 0$.
Using discretization techniques on (25) the problem can be converted into a problem in matrix form. Approximating the equation (25) by finite-difference scheme on the interval $[0,1]$ with a step size $h = \frac{1}{m+1}$ the following system of equations can be obtained

$$\frac{Y_{j+2} - Y_{j+1} + 6Y_j - 4Y_{j-1} + Y_{j-2}}{h^4} - \lambda Y_j + \frac{2\mu(Y_{j+1} - 2Y_j + Y_{j-1})}{h^2} = 0; j := 1:m,$$

$$Y_0 = 0 = Y_{m+1}, Y_1 = Y_{-1}, Y_{m+2} = Y_m.$$

In matrix form

$$\mathbb{P}(\lambda,\mu)u := (M_0 + \lambda M_1 + \mu M_2)u = 0. \tag{25}$$

Here $u = (Y_1, Y_2, Y_3, \ldots, Y_m) \in R^m$. M_1 is a diagonal matrix, M_0 and M_2 are respectively the penta and tridiagonal matrices of order $m \times m$ of the form

$$M_0 = \frac{1}{h^4}\begin{pmatrix} 7 & -4 & 1 & 0 & \ldots & 0 \\ -4 & 6 & -4 & 1 & \ldots & 0 \\ 1 & -4 & 6 & -4 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 7 \end{pmatrix}, M_2 = \frac{1}{h^2}\begin{pmatrix} -1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -2 \end{pmatrix}.$$

The bifurcation points of two algebraic simple eigenvalue curves are in $(-9\pi^4, 5\pi^2)$ and $(-64\pi^4, 10\pi^2)$.

4 Conclusion

Calculation of partial derivatives of matrix determinant involved in (10) is an important task to solve the system (10) to get bifurcation points. Moreover, to use numerical method in the system (11), the coefficient matrix $A(\lambda_m, \mu_m)$ need to be nonsingular. It follows from Theorem 2 that the nonsingularity of the matrix in the equation (22) at simple eigenvalue implies that the coefficient matrix is also nonsingular.

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References

Atkinson, F.V. (1968). Multiparameter spectral theory. *Bulletin of the American Mathematical Society*, *74*, 1-27.

Atkinson, F.V. (1972). *Multiparameter Eigenvalue Problems.* Academic Press, New York.

Attia, L., Barton, M.O. (2021). Shapley–Snow Kernels, Multiparameter Eigenvalue Problems, and Stochastic Games. *Mathematics of Operations Research*, *46*(3), 1181-1202.

Dong, B., Yu, B. & Yu, Y. (2016). A Homotopy method for finding all solutions of a Multiparameter eigenvalue problem. *SIAM Journal on Matrix Analysis and Applications*, *37*(2), 550-571.

Ghosh, A., Alam, R. (2018). Sensitivity and backward perturbation analysis of Multiparameter Eigenvalue Problems. *SIAM Journal on Matrix Analysis and Applications 39*(4), 1750-1775.

Gil, M. (2016). Bounds for the spectrum of a two-parameter matrix eigenvalue problem. *Linear Algebra and its Applications*, *498*(1), 201-218.

Hochstenbach, M.E., Kosir, T. & Plestenjak, B. (2005). A Jacobi-Davidson type method for the nonsingular two-parameter eigenvalue problem. *SIAM Journal on Matrix Analysis and Applications*, *26*, 477-497.

Hochstenbach, M.E., Plestenjak, B. (2008). Harmonic rayleigh ritz extraction for the multiparameter eigenvalue problem. *Electronic Transactions on Numerical Analysis*, *29*, 81-96.

Ji, X. (1992). A two-dimensional bisection method for solving two-parameter eigenvalue problems. *SIAM Journal on Matrix Analysis and Applications*, *13*(4), 1085-1093.

Khlobystov, V.V., Podlevskyi, B.M. (2009). Numerical method for finding bifurcation points of the linear two-parameter eigenvalue problem. *Computational Methods in Applied Mathematics*, *9*(4), 332-338.

Košir, T. (1994). Finite-dimensional multiparameter spectral theory: The nonderogatory case. *Linear Algebra and its Applications*, *212*, 45-70.

Plestenjak, B. (2000). A continuation method for a right definite two-parameter eigenvalue problem. *SIAM Journal on Matrix Analysis and Applications*, *21*, 1163-1184.

Plestenjak, B. (2017). Minimal determinantal representations of bivariate polynomials. *Linear Algebra and its Applications*, *532*, 550-569.

Podlevskyi, B.M., Khlobystov, V.V. (2010). On one approach to finding eigenvalue curves of linear two-parameter spectral problems. *Journal of Mathematical Sciences*, *167*(1), 96-106.

Podlevskyi, B.M. (2007). On certain two-sided analogues of Newton's method for solving nonlinear eigenvalue problems. *Computational Mathematics and Mathematical Physics*, *47*(11), 1745-1755.

Podlevskyi, B.M. (2008). Newton's method as a tool for finding the eigenvalues of certain two-parameter (multiparameter) spectral problems. *Computational Mathematics and Mathematical Physics*, *48*(12), 2107-2112.

Pons, A., Gutschmidt, S. (2017). Multiparameter solution methods for semi-structured Aeroelastic flutter problems. *AIAA journal, 55*(10), 3530-3538.

Ringh, E., Jarlebring, E. (2021). Nonlinearizing two-parameter eigenvalue problem. *SIAM Journal on Matrix Analysis and Applications*, *42*(2), 775-799.

Rodriguez, J.I., Du, J.H, You, Y. & Lim, L.H. (2018). *Fiber product homotopy method for multiparameter eigenvalue problems.* Priprint, Cornell University.

Sleeman, B.D. (1978). *Multiparameter Spectral Theory in Hilbert Space.* (Research Notes in Mathematics vol 22). Pitman, London.

Volkmer, H. (1988). *Multiparameter Eigenvalue Problems and Expansion Theorem.* Lecture Notes in Mathematics 1356, Springer-Verlag, Berlin, New York.